Supplemental note for Week 4 Part 1

ver. 20170416-01

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1 Average and variance of the cumulative impulse $\Delta\mathbf{W}_i$

Let us start with the Langevin equation

$$m\frac{d\mathbf{V}(t)}{dt} = -\zeta\mathbf{V}(t) + \mathbf{F}(t), \tag{F2}$$

where the random force $\mathbf{F}(t)$ satisfies the following conditions

$$\langle\mathbf{F}(t)\rangle = \mathbf{0} \tag{F3}$$

$$\langle\mathbf{F}(t)\mathbf{F}(0)\rangle = 2k_BT\zeta\mathbf{I}\delta(t), \tag{F4}$$

with $\mathbf{0} \equiv (0,0,0)$ and $\mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We now discretize the time t using an increment Δt, such that $t_i \equiv i\Delta t$, and define the cumulative impulse during the interval $t_i \le t \le t_{i+1} = t_i + \Delta t$, as

$$\Delta\mathbf{W}_i \equiv \int_{t_i}^{t_{i+1}} dt\mathbf{F}(t). \tag{F8}$$

From Eq.(F3), it is straightforward to show

$$\langle\Delta\mathbf{W}_i\rangle = \int_{t_i}^{t_{i+1}} dt\langle\mathbf{F}(t)\rangle = \mathbf{0}. \tag{F10}$$

Also from Eq.(F4), for $j \ne i$

$$\begin{aligned} \langle\Delta\mathbf{W}_i\Delta\mathbf{W}_{j\ne i}\rangle &= \int_{t_i}^{t_{i+1}} dt \int_{t_j}^{t_{j+1}} dt' \langle\mathbf{F}(t)\mathbf{F}(t')\rangle & (1) \\ &= 2k_BT\zeta\mathbf{I}\int_{t_i}^{t_{i+1}} dt \int_{t_j}^{t_{j+1}} dt'\delta(t-t') & (2) \\ &= \mathbf{O}, & (3) \end{aligned}$$

where $\mathbf{O} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

For $j = i$

$$\begin{aligned} \langle\Delta\mathbf{W}_i\Delta\mathbf{W}_i\rangle &= \int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} dt' \langle\mathbf{F}(t)\mathbf{F}(t')\rangle & (4) \\ &= 2k_BT\zeta\mathbf{I}\int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} dt'\delta(t-t') & (5) \\ &= 2k_BT\zeta\Delta t\mathbf{I}. & (6) \end{aligned}$$

Combining Eqs.(3) and (6), we obtain

$$\langle\Delta\mathbf{W}_i\Delta\mathbf{W}_j\rangle = 2k_BT\zeta\Delta t\mathbf{I}\delta_{ij}. \tag{F11}$$

2 Distribution of $\Delta\mathbf{W}_i$

Here we further divide Δt into n segments ($n \gg 1$) of a very small time span ϵ, *i.e.*, $\Delta t \equiv n\epsilon$, and define a new cumulative impulse over ϵ

$$\mathbf{W}_i^m \equiv \int_{t_i+(m-1)\epsilon}^{t_i+m\epsilon} dt\mathbf{F}(t), \tag{7}$$

where $1 \leq m \leq n$.
Repeating the same procedure performed in the previous section, the following conditions are derived.

$$\langle \mathbf{W}_i^m \rangle = \mathbf{0} \tag{8}$$
$$\langle \mathbf{W}_i^m \mathbf{W}_j^l \rangle = 2k_B T\zeta\epsilon\mathbf{I}\delta_{ij}\delta_{ml} \tag{9}$$

Eqs.(8) and (9) show that the mean and variance of the random numbers $W_{\alpha,i}^m$ ($\alpha \in x, y, z$) are zero and $2k_BT\zeta\epsilon$, respectively.
From Eq.(F8) and (7), we should notice that

$$\Delta\mathbf{W}_i = \mathbf{W}_i^1 + \mathbf{W}_i^2 + \cdots + \mathbf{W}_i^n. \tag{10}$$

Therefore, from the central limit theorem Eqs.(D7)-(D9) introduced in Part 3 of Week 2, one realizes that the $\Delta W_{\alpha,i}$ should be drawn from a *Gaussian* distribution, with average and variance equal to zero and $2k_BT\zeta\Delta t$, respectively, regardless of the distribution of the $W_{\alpha,i}^m$.