Supplemental note for Week 3 Part 2

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1 Derivation of the mean square displacement

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\begin{aligned}
\left\langle |\Delta \mathbf{R}(t)|^2 \right\rangle &= \left\langle \int_0^t dt_1 \mathbf{V}(t_1) \cdot \int_0^t dt_2 \mathbf{V}(t_2) \right\rangle && (1)\\
&= \int_0^t dt_1 \int_0^t dt_2 \left\langle \mathbf{V}(t_1) \cdot \mathbf{V}(t_2) \right\rangle && (2)\\
&= \int_0^t dt_1 \int_0^t dt_2 \frac{3\tilde{D}}{\zeta m} \exp\left(-\frac{\zeta}{m} |t_2 - t_1| \right) && (3)\\
&= 2\int_0^t dt_1 \int_{t_1}^t dt_2 \frac{3\tilde{D}}{\zeta m} \exp\left(-\frac{\zeta}{m} (t_2 - t_1) \right) && (4)\\
&= \frac{6\tilde{D}}{\zeta m} \int_0^t dt_1 \left[\exp\left(\frac{\zeta}{m} t_1 \right) \int_{t_1}^t dt_2 \exp\left(-\frac{\zeta}{m} t_2 \right) \right] && (5)\\
&= \frac{6\tilde{D}}{\zeta m} \int_0^t dt_1 \left[\exp\left(\frac{\zeta}{m} t_1 \right) \left(-\frac{m}{\zeta} \left(\exp\left(-\frac{\zeta}{m} t \right) - \exp\left(-\frac{\zeta}{m} t_1 \right) \right) \right) \right] && (6)\\
&= \frac{6\tilde{D}}{\zeta^2} \int_0^t dt_1 \left[\exp\left(\frac{\zeta}{m} t_1 \right) \left(\exp\left(-\frac{\zeta}{m} t_1 \right) - \exp\left(-\frac{\zeta}{m} t \right) \right) \right] && (7)\\
&= \frac{6\tilde{D}}{\zeta^2} \int_0^t dt_1 \left[1 - \exp\left(\frac{\zeta}{m} t_1 \right) \exp\left(-\frac{\zeta}{m} t \right) \right] && (8)\\
&= \frac{6\tilde{D}}{\zeta^2} \left[t - \exp\left(-\frac{\zeta}{m} t \right) \int_0^t dt_1 \exp\left(\frac{\zeta}{m} t_1 \right) \right] && (9)\\
&= \frac{6\tilde{D}}{\zeta^2} \left[t - \exp\left(-\frac{\zeta}{m} t \right) \frac{m}{\zeta} \left(\exp\left(\frac{\zeta}{m} t \right) - 1 \right) \right] && (10)\\
&= \frac{6\tilde{D}}{\zeta^2} \left[t - \frac{m}{\zeta} + \frac{m}{\zeta} \exp\left(-\frac{\zeta}{m} t \right) \right] && (11)\\
&\simeq \frac{6\tilde{D}}{\zeta^2} t \qquad (t \to \infty) && (12)
\end{aligned}
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