

Supplemental note for Week 2 Part 3

ver. 20170418-01

Ryoichi Yamamoto

1 Binomial distribution \rightarrow Normal distribution

Start from Eq.(C6) shown below.

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \quad (1)$$

$$\ln P(n) = \ln M! - \ln n! - \ln(M-n)! + n \ln p + (M-n) \ln(1-p) \quad (2)$$

$$= M(\ln M - 1) - n(\ln n - 1) - (M-n)(\ln(M-n) - 1) \quad (3)$$

$$+ n \ln p + (M-n) \ln(1-p) \quad (4)$$

Here we used Stirling's approximation valid for large n .

$$\ln n! \simeq n(\ln n - 1) \quad (5)$$

Take the 1st derivative of the above equation in terms of n .

$$\frac{d \ln P}{dn} = -(\ln n + 1) + (\ln(M-n) + 1) + \ln p - \ln(1-p) \quad (6)$$

$$= -\ln n + \ln(M-n) + \ln p - \ln(1-p) \quad (7)$$

$$= \ln \left[\frac{M-n}{n} \right] - \ln \left[\frac{1-p}{p} \right] \quad (8)$$

When n and M are both large, the peak in $P(n)$ and also in $\ln P(n)$ is very sharp around the mean value $n = \langle N \rangle$, and thus

$$\left. \frac{d \ln P}{dn} \right|_{n=\langle N \rangle} = \ln \left[\frac{M - \langle N \rangle}{\langle N \rangle} \right] - \ln \left[\frac{1-p}{p} \right] = 0 \quad (9)$$

$$\therefore \frac{M - \langle N \rangle}{\langle N \rangle} = \frac{1-p}{p} \quad (10)$$

$$\langle N \rangle = Mp. \quad (11)$$

Consider a Taylor expansion of $\ln P(n)$ around the mean $\langle N \rangle$ by defining $n = \langle N \rangle + \delta n$.

$$\ln P(n) = \ln(\langle N \rangle) + \left. \frac{d \ln P}{dn} \right|_{n=\langle N \rangle} \delta n + \frac{1}{2} \left. \frac{d^2 \ln P}{dn^2} \right|_{n=\langle N \rangle} \delta n^2 + \dots \quad (12)$$

From Eq.(7),

$$\frac{d^2 \ln P}{dn^2} = -\frac{1}{n} - \frac{1}{M-n} \quad (13)$$

$$\left. \frac{d^2 \ln P}{dn^2} \right|_{n=\langle N \rangle} = -\frac{1}{Mp} - \frac{1}{M-Mp} = -\frac{1}{Mp(1-p)} = -\sigma^{-2}. \quad (14)$$

Terminate Eq.(12) at the 2nd order in terms of δn , and use Eqs.(9) and (14).

$$P(n) = \text{const.} \times \exp\left(-\frac{\delta n^2}{2\sigma^2}\right) \quad (15)$$

Determine *const.* so that that $\int_{-\infty}^{\infty} P(n)dn = 1$, we finally obtain the following normal distribution function with $\langle N \rangle = Mp$ and $\sigma^2 = Mp(1-p)$.

$$P(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(n - \langle N \rangle)^2}{2\sigma^2}\right) \quad (16)$$

2 Binomial distribution \rightarrow Poisson distribution

We now consider the limit of $M \rightarrow \infty$ while $\langle N \rangle = Mp = a$ remains constant. Notice that the following approximations hold.

$$M - n \simeq M \quad (17)$$

$$\frac{M!}{(M-n)!} = M(M-1)\cdots(M-n+1) \simeq M^n \quad (18)$$

Again start from Eq.(C6), we can derive Poisson distribution as shown below with $\langle N \rangle = Mp = a$ and $\sigma^2 = Mp(1-p) \simeq a$.

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \quad (19)$$

$$\simeq \frac{1}{n!} M^n \left(\frac{a}{M}\right)^n \left(1 - \frac{a}{M}\right)^M \quad (20)$$

$$\simeq \frac{a^n e^{-a}}{n!} \quad (21)$$

Here we used

$$\lim_{M \rightarrow \infty} \left(1 - \frac{a}{M}\right)^M = e^{-a}. \quad (22)$$