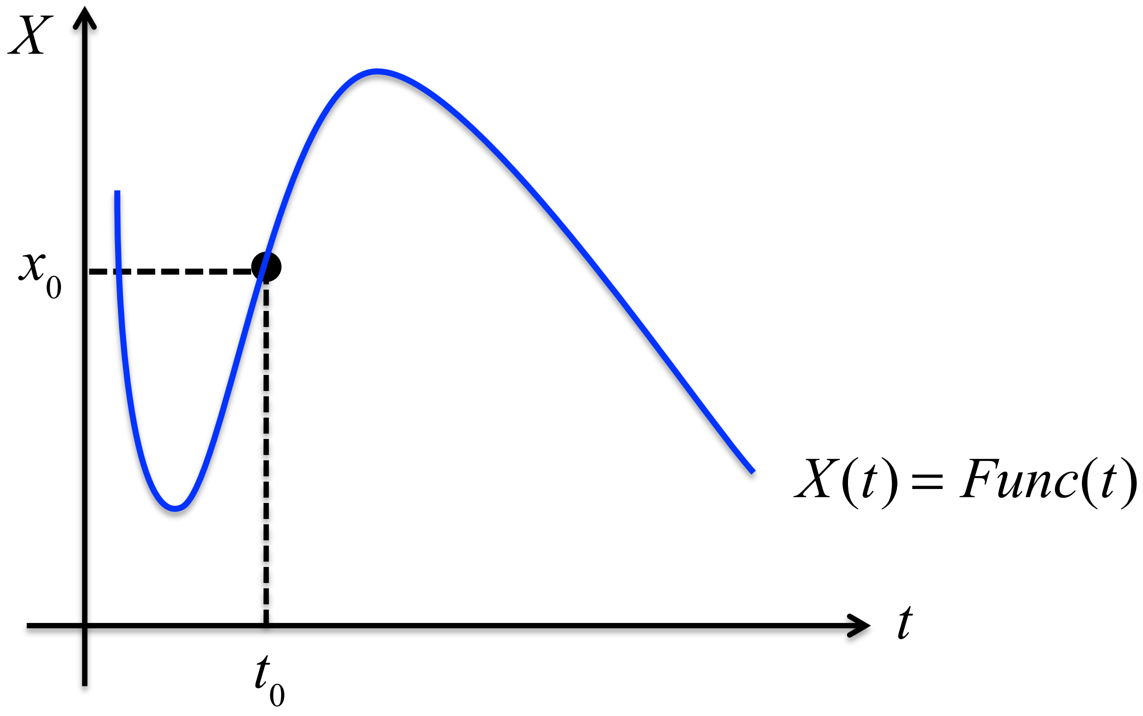
Brownian motion 1: basic theories

Basic knowledge of stochastic process

Stochastic process

A deterministic process:

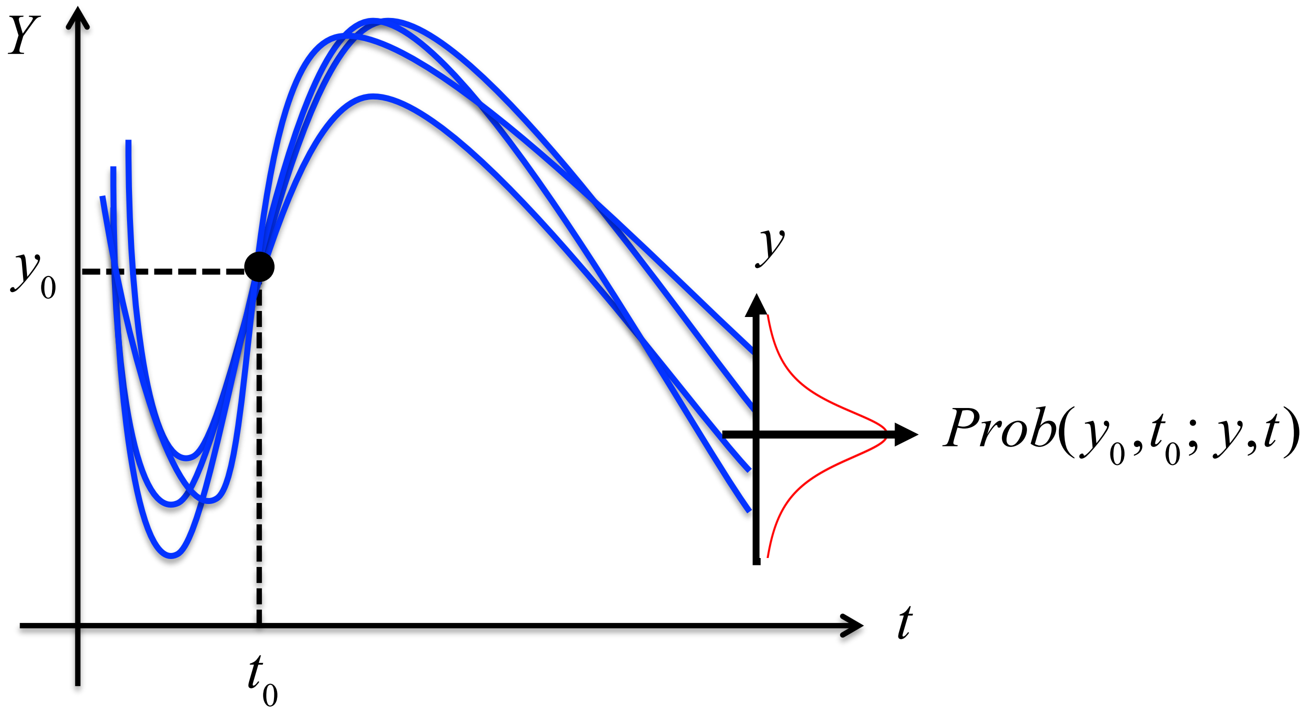
$$X(t) = Func(t)$$



Stochastic process

A stochastic process:

$$Y(t) \neq Func(t) \quad \rightarrow \quad Prob(y_0, t_0; y, t)$$



Stochastic process

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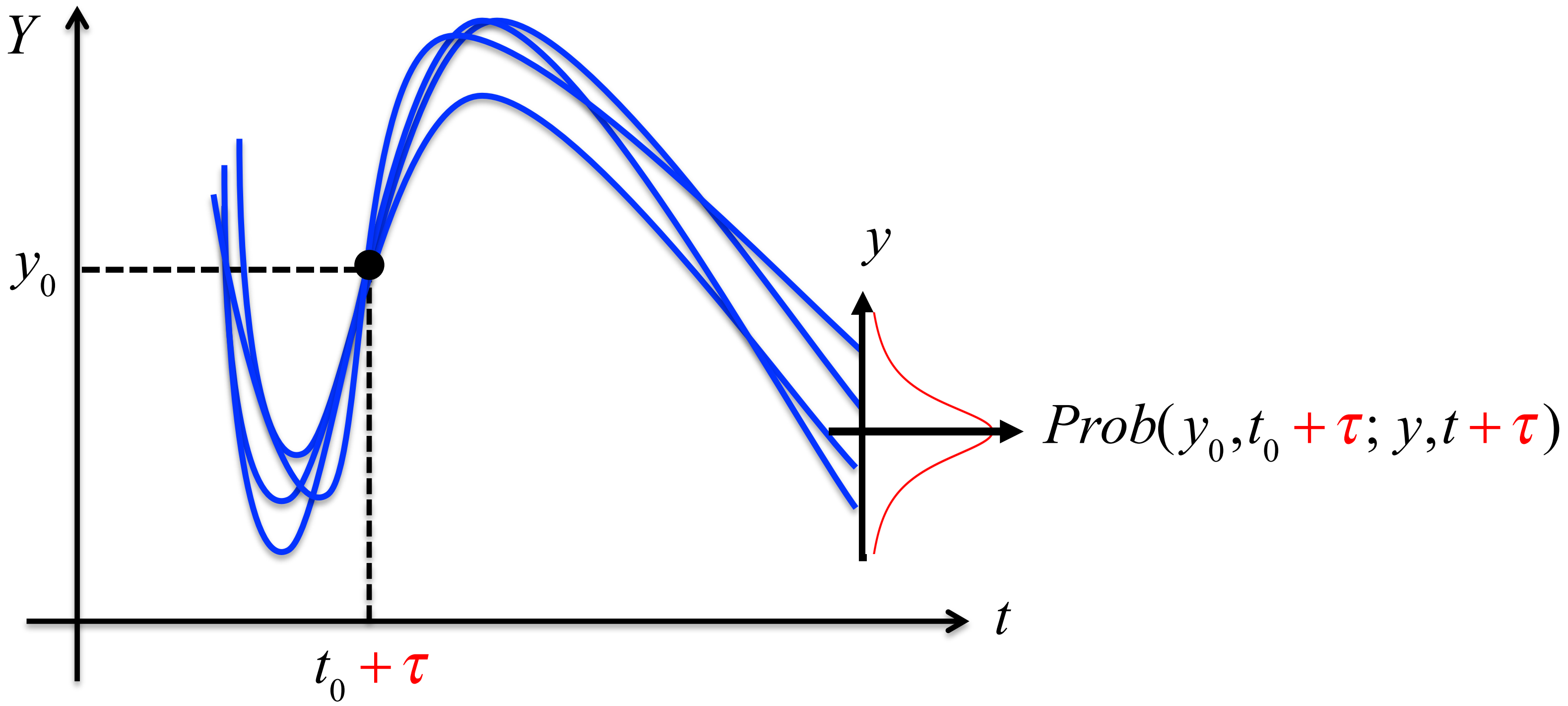


A steady stochastic process:

$$Prob(y_0, t_0 + \tau; y, t + \tau) = Prob(y_0, t_0; y, t)$$

Stochastic process

Consider a steady stochastic process $Y(t)$ with its mean $\langle Y(t) \rangle = 0$ and define Fourier transformation

$$\tilde{Y}_T(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} Y_T(t) \qquad (1)$$

and inverse Fourier transformation

$$Y_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \tilde{Y}_T(\omega) \qquad (2)$$

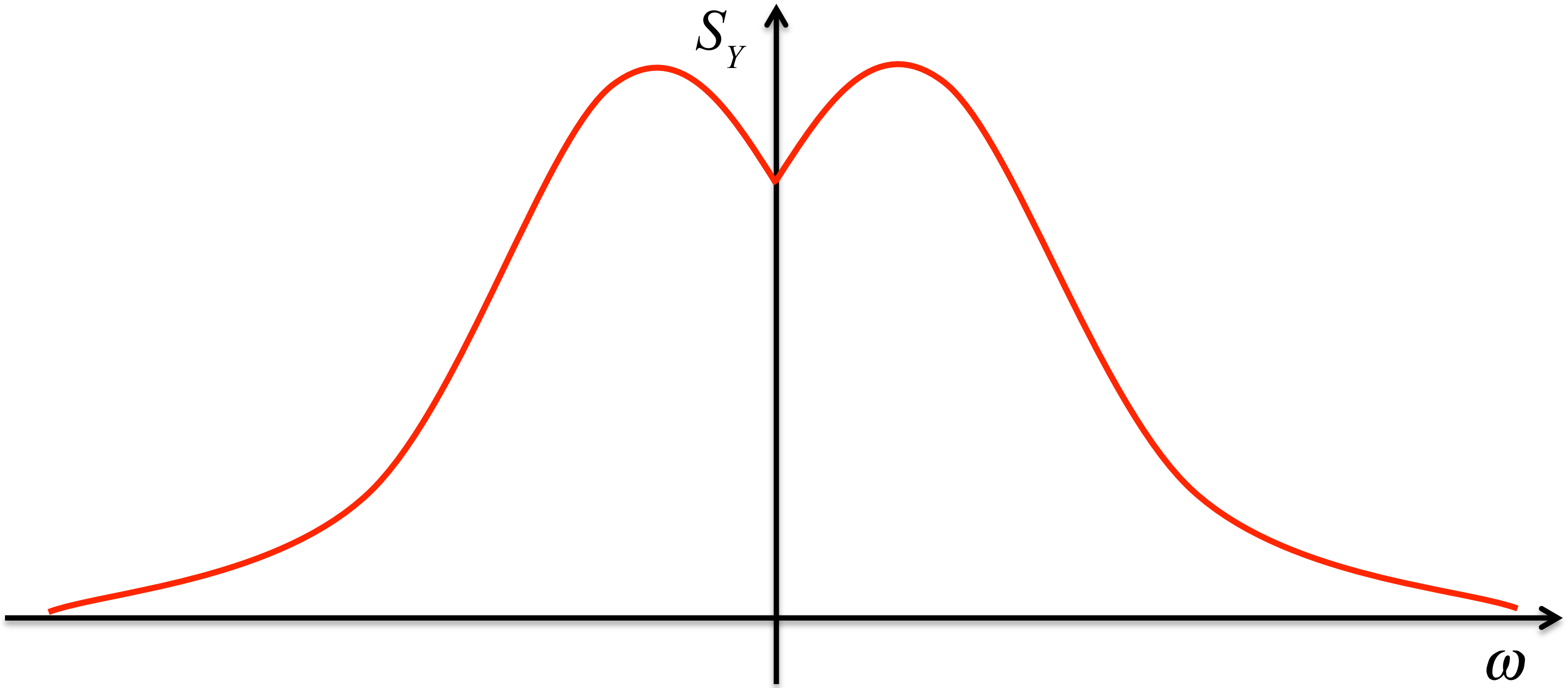
using

$$\begin{aligned} Y_T(t) &= Y(t) \quad \left(|t| \le T/2\right) \\ Y_T(t) &= 0 \quad \left(|t| > T/2\right) \end{aligned} \qquad (3)$$

Stochastic process

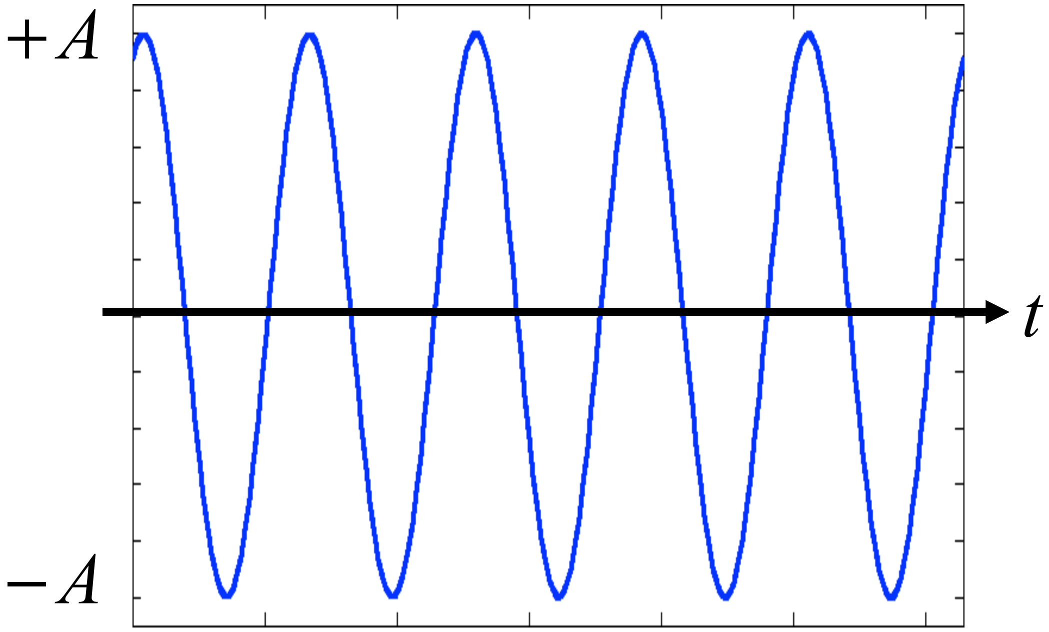
Spectral density / Power spectrum

$$S_Y(\omega) \equiv \lim_{T\to\infty} \frac{1}{T} \left| \tilde{Y}_T(\omega) \right|^2 \qquad (4)$$

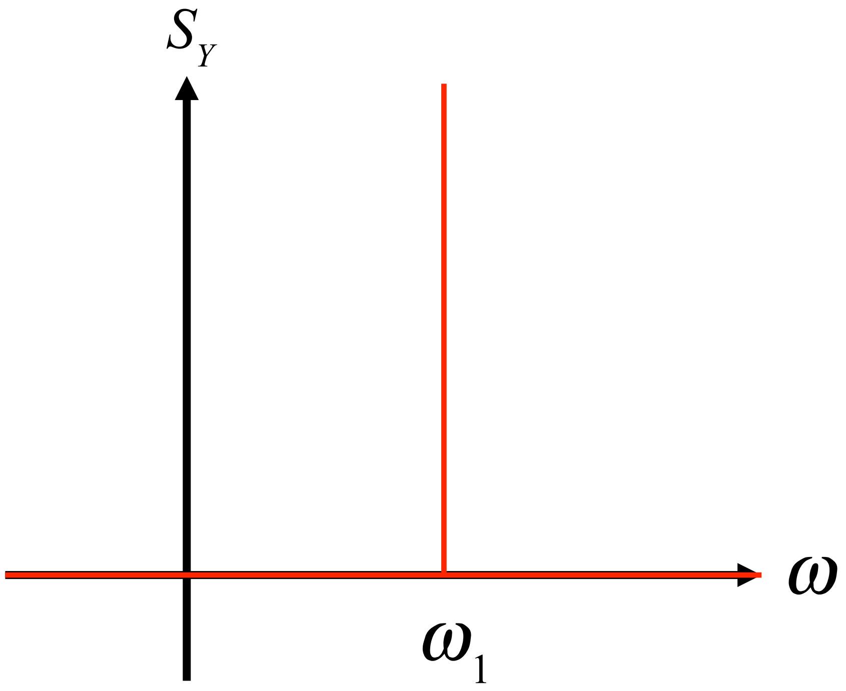


Stochastic process

Case 1: Single cosine wave



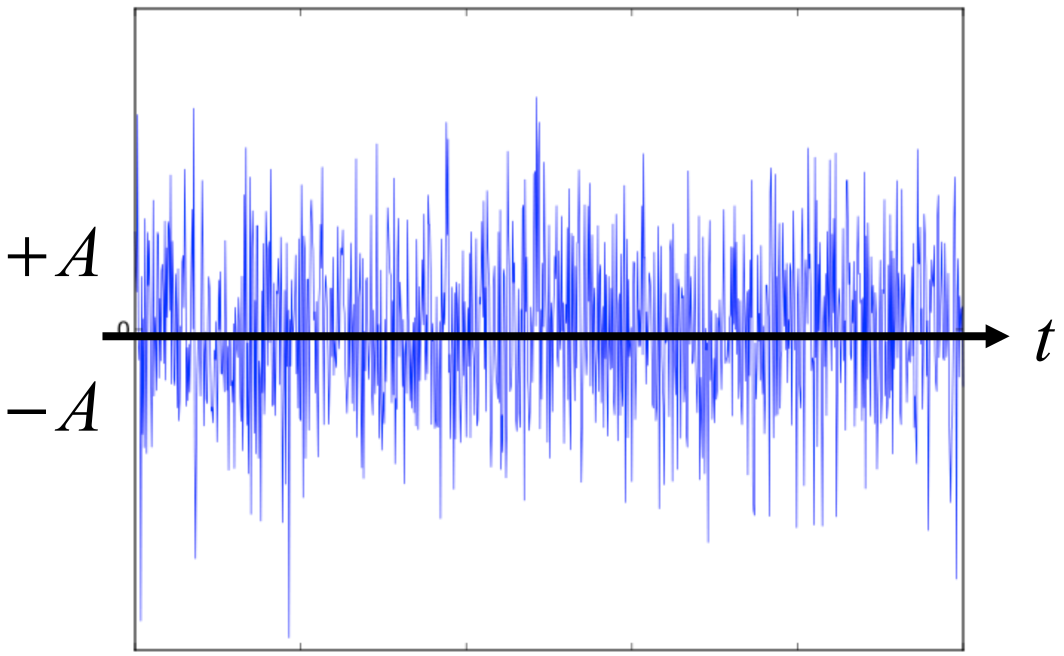
$$Y(t) = A\cos(\omega_1 t) \qquad (5)$$



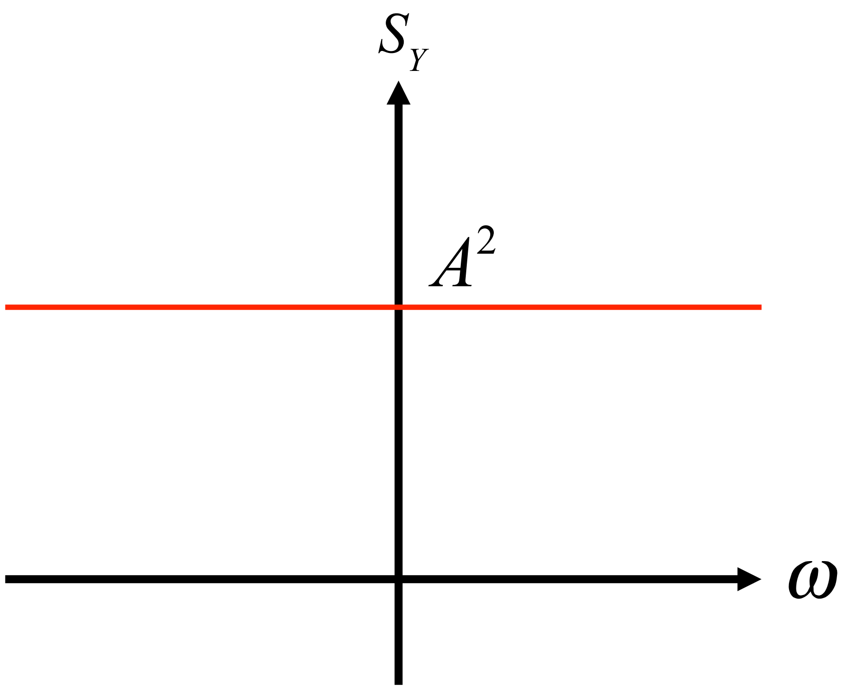
$$S_Y(\omega) = A^2\delta(\omega - \omega_1) \qquad (6)$$

Stochastic process

Case 2: White noise



$$Y(t) = A\xi(t) \qquad (7)$$

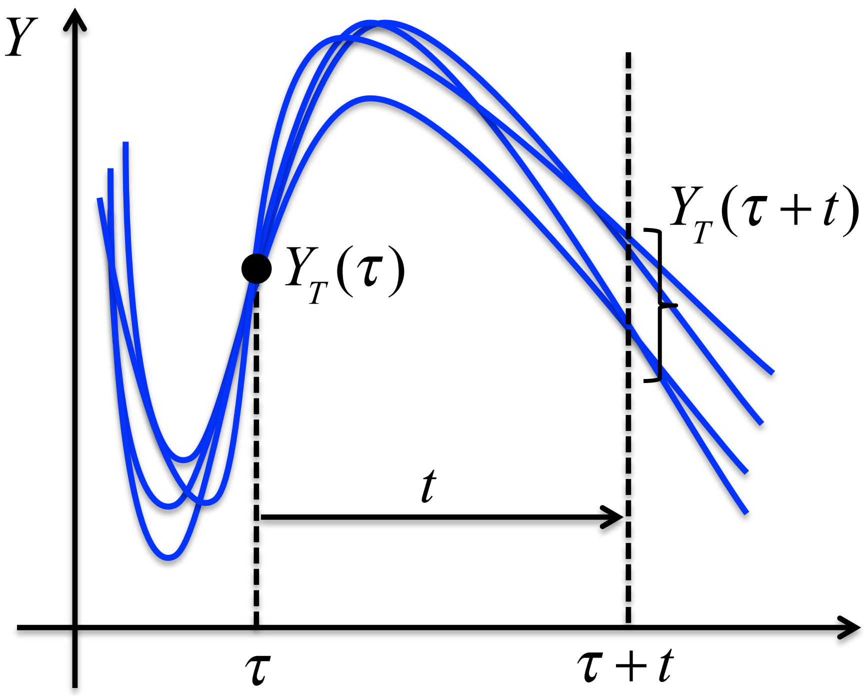


$$S_Y(\omega) = A^2 \qquad (8)$$

Stochastic process

Auto-correlation function

$$\varphi_Y(t) \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau\, Y_T(\tau) Y_T(\tau + t) \equiv \langle Y(\tau) Y(\tau + t) \rangle_\tau \tag{9}$$



Auto-correlation function

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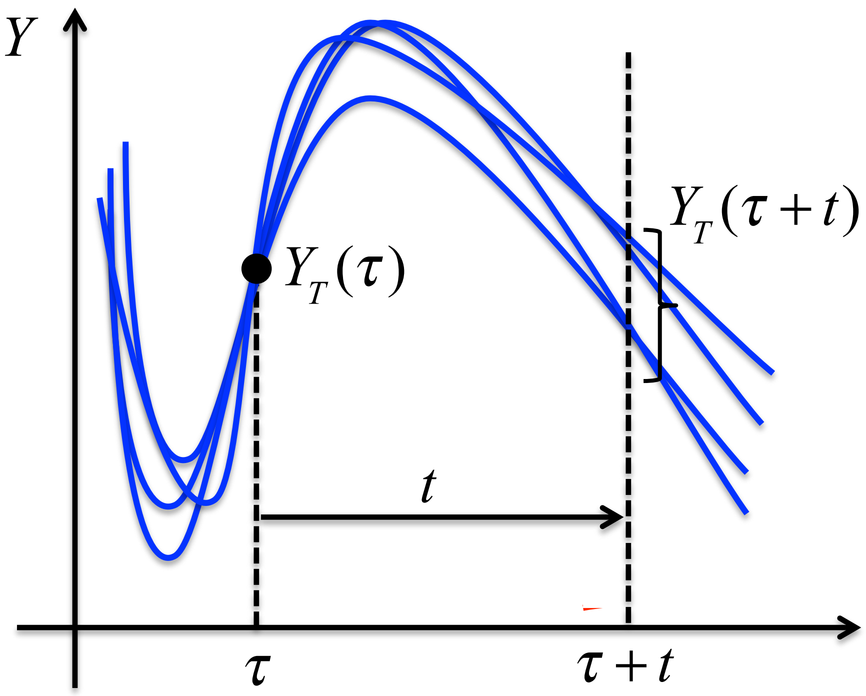
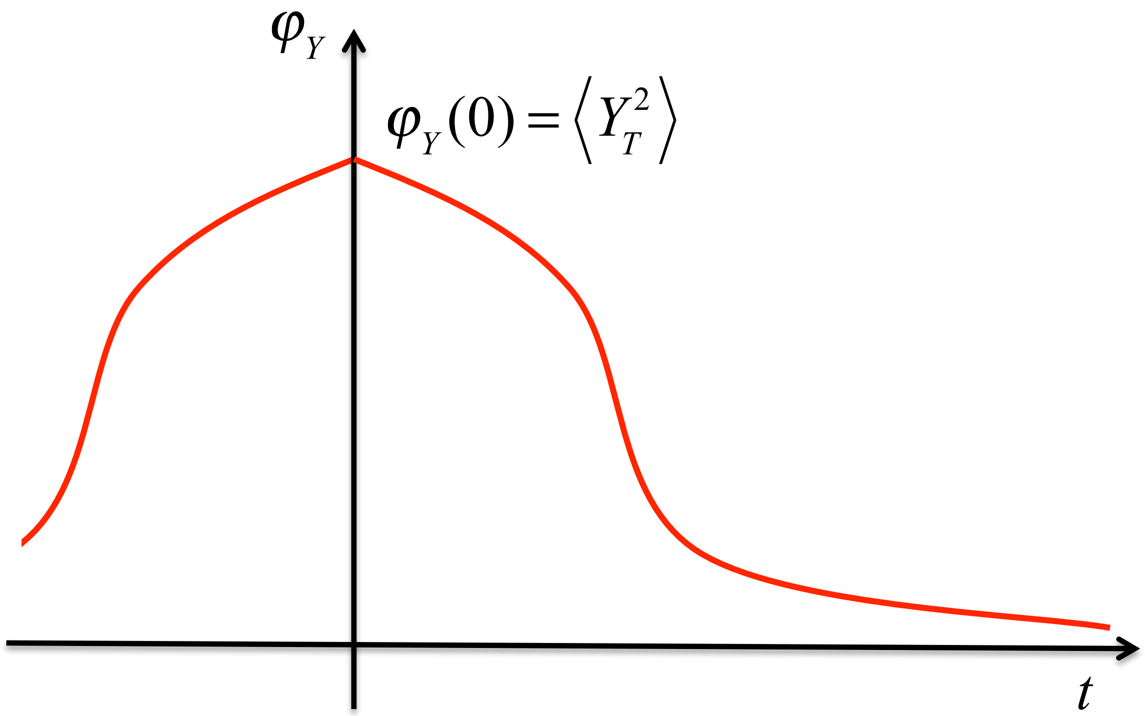
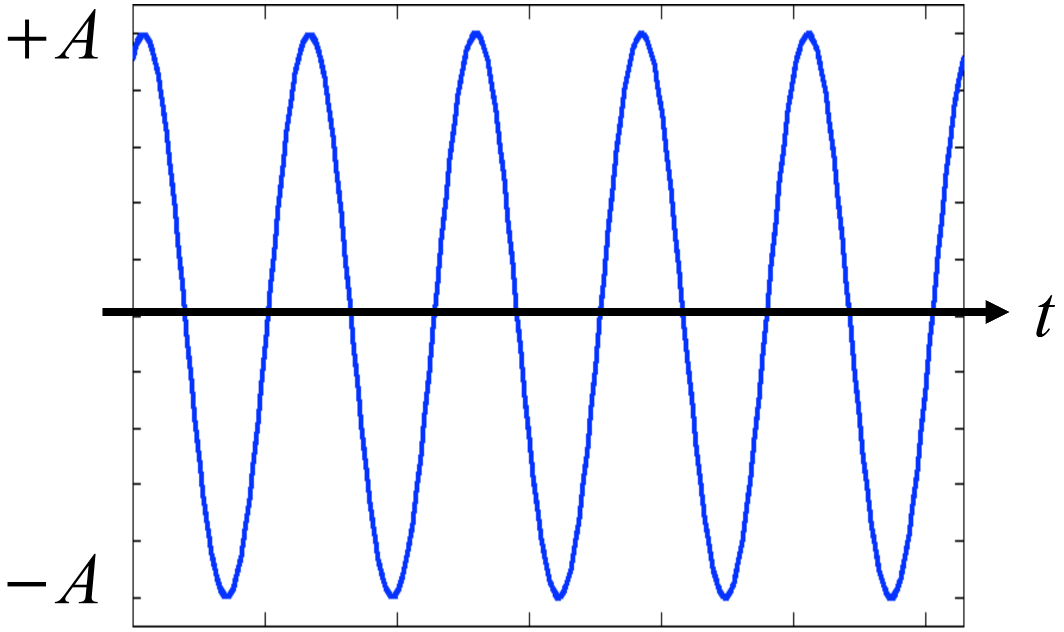


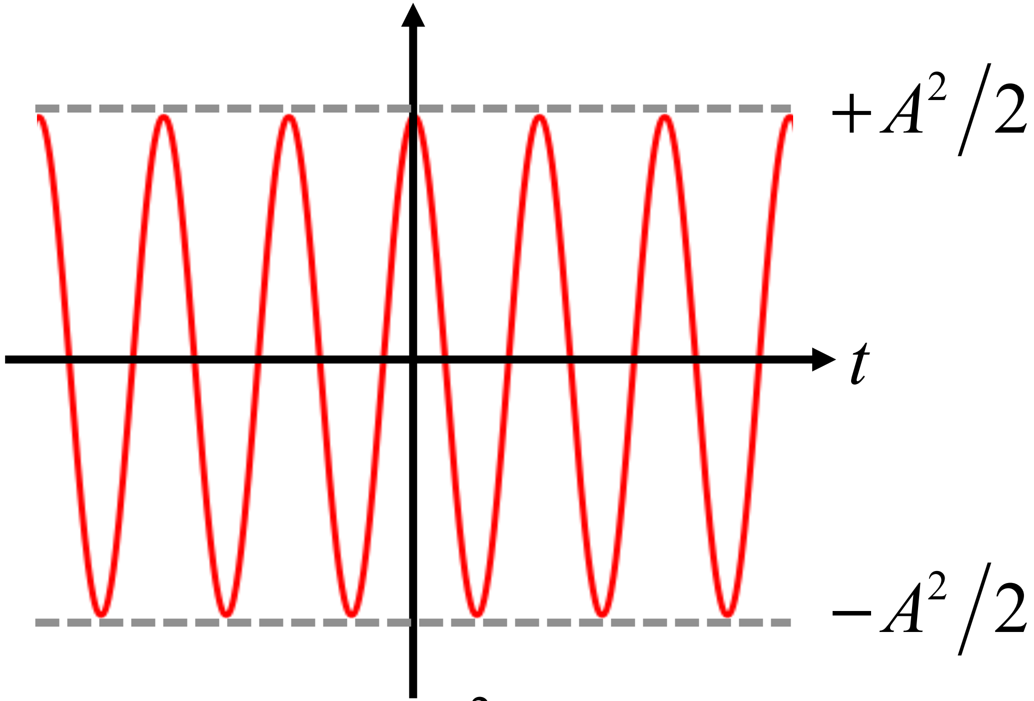


Stochastic process

Case 1: Single cosine wave



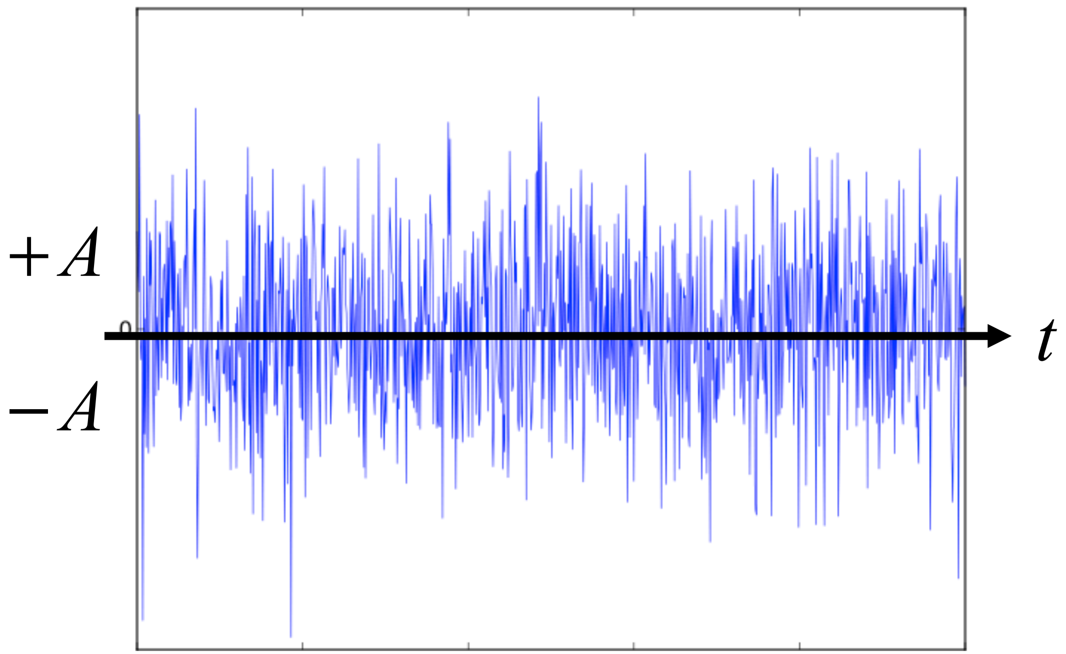
$$Y(t) = A\cos(\omega_1 t) \qquad (10)$$



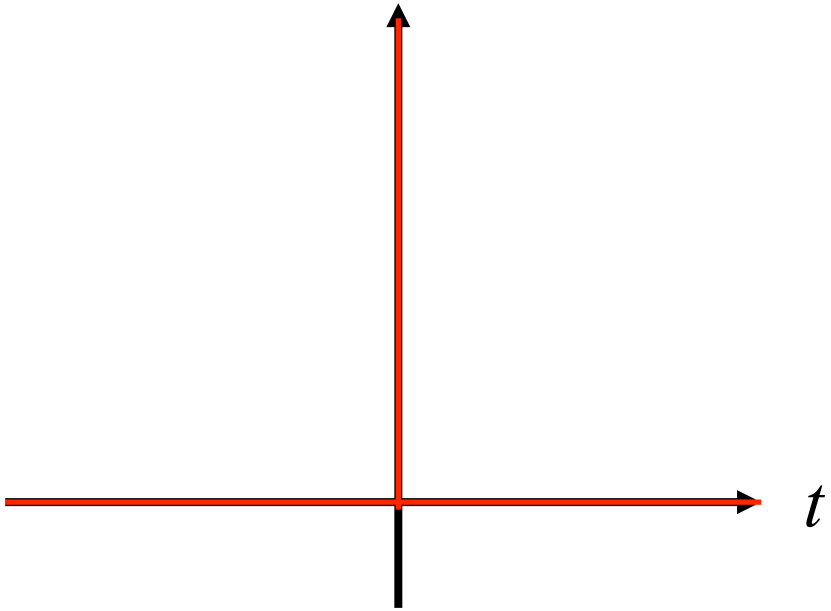
$$\varphi_Y(t) = \frac{A^2}{2}\cos(\omega_1 t) \qquad (11)$$

Stochastic process

Case 2: White noise



$$Y(t) = A\xi(t) \qquad (12)$$



$$\varphi_Y(\omega) = A^2\delta(t) \qquad (13)$$

Stochastic process

From Eq.(9),

$$\varphi_Y(t) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau \left[Y_T(\tau) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau+t)} \tilde{Y}_T(\omega) \right] \right]$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} d\omega \left[e^{-i\omega t} \tilde{Y}_T(\omega) \int_{-\infty}^{\infty} d\tau \left[e^{-i\omega\tau} Y_T(\tau) \right] \right]$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} d\omega \left[e^{-i\omega t} \tilde{Y}_T(\omega) \tilde{Y}_T^*(\omega) \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \left[\lim_{T\to\infty} \frac{1}{T} \left| \tilde{Y}_T(\omega) \right|^2 \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} S_Y(\omega) \qquad (14)$$

Stochastic process

And also,

$$S_Y(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \varphi_Y(t) \tag{15}$$

Wiener-Khintchine theorem:

$$\boxed{\varphi_Y(t) \underset{\text{Fourier Eq.(15)}}{\overset{\text{inverse Fourier Eq.(14)}}{\rightleftarrows}} S_Y(\omega)}$$

Sum rules:

$$\varphi_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, S_Y(\omega) \tag{16}$$

$$S_Y(0) = \int_{-\infty}^{\infty} dt\, \varphi_Y(t) \tag{17}$$

Stochastic process

Supplemental note for the derivation of Eq. (11)

$$\varphi_Y(t) = \lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \cos(\omega_1\tau)\cos(\omega_1(\tau+t))$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin\left(\omega_1\tau+\frac{\pi}{2}\right)\sin\left(\omega_1(\tau+t)+\frac{\pi}{2}\right)$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin\left(\omega_1\tau+\frac{\pi}{2}\right)\left[\sin\left(\omega_1\tau+\frac{\pi}{2}\right)\cos(\omega_1 t)+\cos\left(\omega_1\tau+\frac{\pi}{2}\right)\sin(\omega_1 t)\right]$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left[\sin^2\left(\omega_1\tau+\frac{\pi}{2}\right)\cos(\omega_1 t)+\sin\left(\omega_1\tau+\frac{\pi}{2}\right)\cos\left(\omega_1\tau+\frac{\pi}{2}\right)\sin(\omega_1 t)\right]$$

$$= \cos(\omega_1 t)\lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin^2\left(\omega_1\tau+\frac{\pi}{2}\right)+\sin(\omega_1 t)\lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin\left(\omega_1\tau+\frac{\pi}{2}\right)\cos\left(\omega_1\tau+\frac{\pi}{2}\right)$$

$$= \frac{A^2}{2}\cos(\omega_1 t)+0 \quad \cdots \quad \text{Eq.}(11)$$