Lab1 Seminar
2018/02/28

Introduction to Econophysics

From Brownian to Black-Scholes and beyond

Ryoichi Yamamoto

Economy of gambling

Gambling	Return rate
• Japanese LOTO / TOTO	45 % / 50 %
• Japanese horse/bike/boat race	75 %
• Pachinko	85-90 % (average)
• Roulette	94.7 % (Las Vegas)
• Blackjack	98-99 % (basic strategy)
• Blackjack	> 100 % (card counting)
• Financial investment	> 100 %

Economy of gambling

Trailer of the movie “21” (2008) inspired by the true story of MIT Blackjack Team

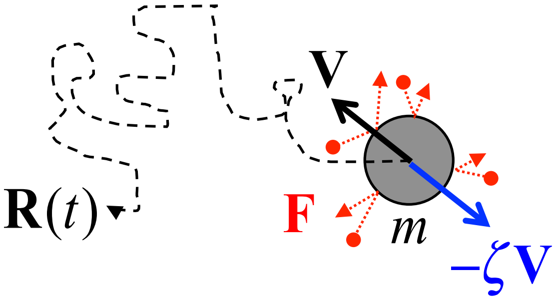
Point	Cards
+1	2, 3, 4, 5, 6
0	7, 8, 9
-1	A, 10, J, Q, K

Count the total point of opened cards during the game.

more + → more favorable
more - → less favorable

Brownian motion

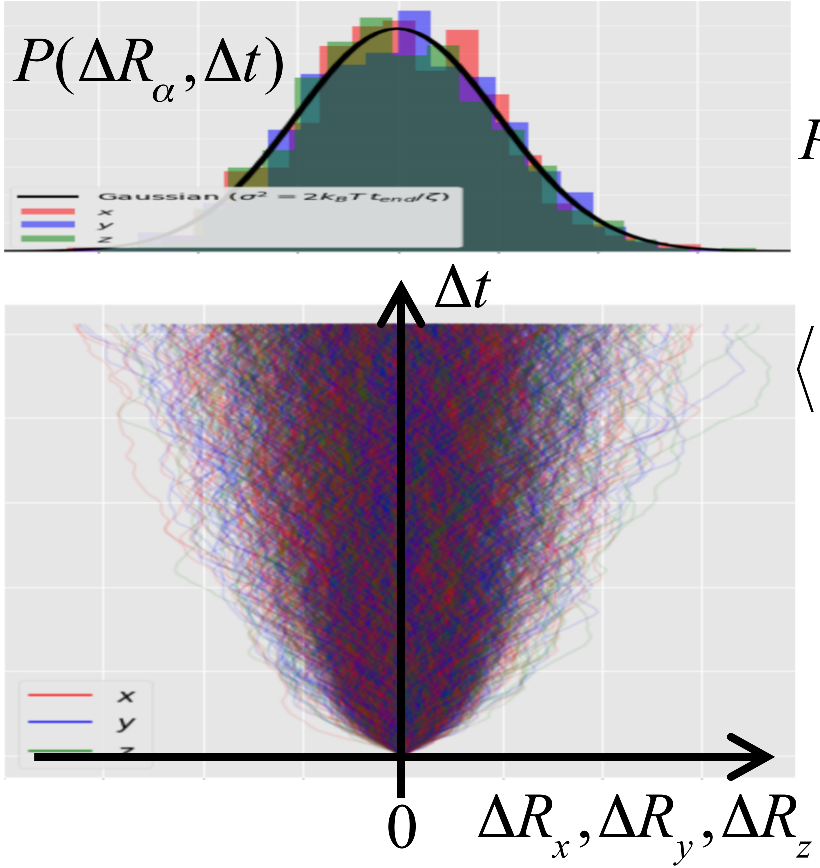
Particle radius:	a
Particle mass:	m
Solvent viscosity:	η
Friction constant:	$\zeta = 6\pi\eta a$
Particle position:	$\mathbf{R}(t)$
Particle velocity:	$\mathbf{V}(t) = d\mathbf{R}/dt$
Friction force:	$-\zeta\mathbf{V}(t)$
Random force:	$\mathbf{F}(t)$



Langevin Equation:

$$m\frac{d\mathbf{V}}{dt} = -\zeta\mathbf{V} + \mathbf{F}$$

Brownian motion



$$P(\Delta R_\alpha, \Delta t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(\Delta R_\alpha - \langle \Delta R_\alpha \rangle\right)^2}{2\sigma^2} \right]$$

$$\langle \Delta R_\alpha \rangle = 0, \quad \sigma^2 = \frac{2k_BT}{\zeta}\Delta t \quad (\alpha = x, y, z)$$

$$\Delta R_\alpha(t) = \sqrt{\frac{k_BT}{\zeta}}\, \Delta W(t)$$ Wiener process

where $\langle \Delta W \rangle = 0, \quad \langle \Delta W^2 \rangle = \Delta t$

Ito lemma

$$\lim_{\Delta t \to 0} \Delta W = dW$$

$$\langle dW \rangle = 0$$

$$\langle dW^2 \rangle = dW^2 = dt$$

$$\therefore dW = \sqrt{dt}$$

Generalized Brownian = Ito process

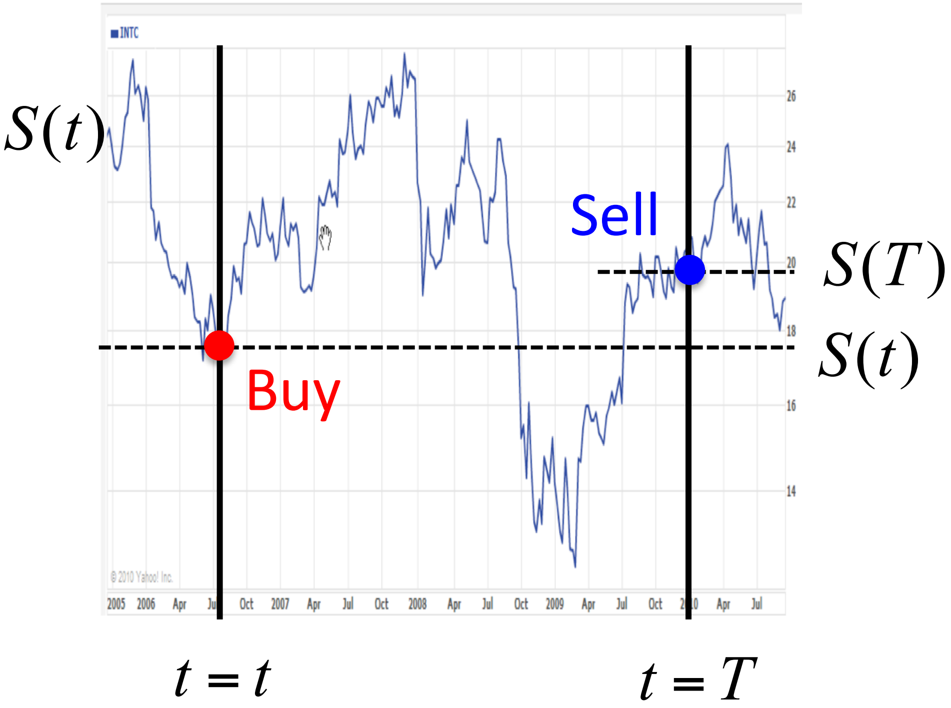
$$dX(t) = a(X(t),t)dt + b(X(t),t)dW$$

Drift Standard dev.

Ito lemma for $f(X(t),t)$

$$df = \left[\frac{\partial f}{\partial X} a(X(t),t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} b^2(X(t),t)\right] dt + \frac{\partial f}{\partial X} b(X(t),t) dW$$

Stock market



Return: $R = S(T) - S(t)$

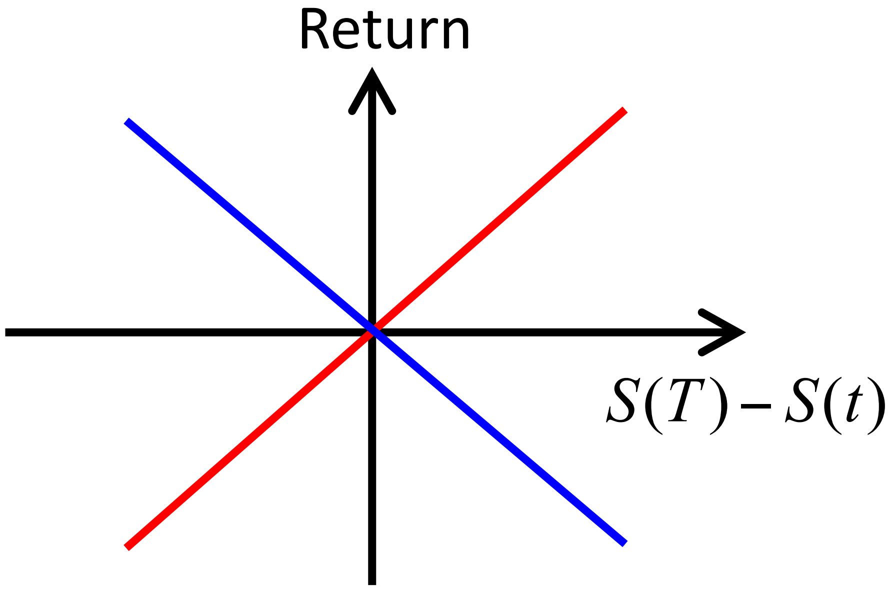
Log return: $lr = \ln \frac{S(T)}{S(t)}$

Return rate: $r = \frac{S(T) - S(t)}{S(t)}$

$$lr = r \quad \text{for} \quad T - t \to 0$$

Stock market

1. Buy
2. Sell



Stock market

Geometric Brownian

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW$$

Drift → Trend   Standard dev. → Volatility

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW$$

$$d \ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW$$

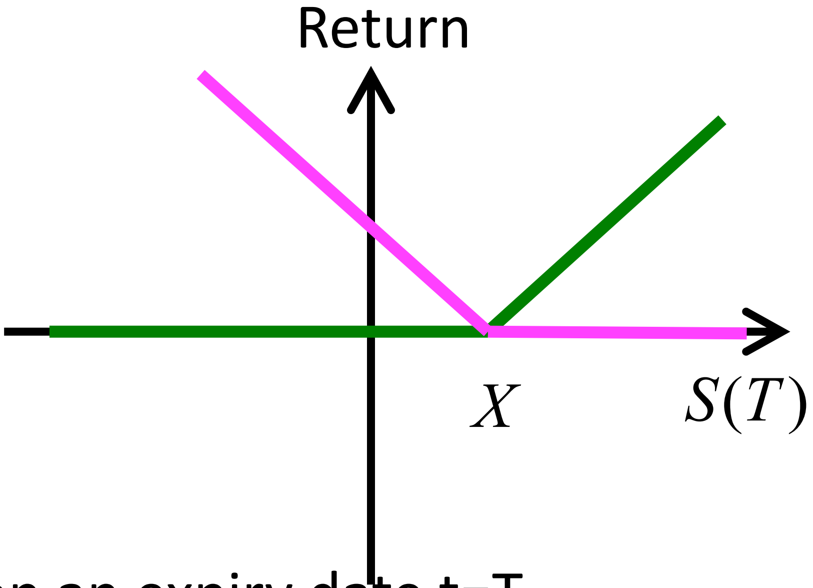
$$\Delta \ln S = \ln S(T) - \ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\, \Delta W$$

(Risk!)

$$\langle \Delta W \rangle = 0$$

$$\langle \Delta W^2 \rangle = \Delta t$$

Financial derivatives



1. Buy
2. Sell
3. Call option (derivative of S)
   =right to buy at a strike price X on an expiry date t=T
4. Put option (derivative of S)
   =right to sell at a strike price X on an expiry date t=T

Financial derivatives

Value of derivative (Premium) = $f(S(t),t)$

Ito lemma for $f(S(t),t)$

$$df = \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right] dt + \frac{\partial f}{\partial S}\sigma S dW$$

Risk!!

Risk hedge

Portfolio (combination of stocks and derivatives)

		#Buy	#Sell
Stock@	$S(t)$	$\frac{\partial f}{\partial S}$	0
Derivative@	$f(S(t),t)$	0	1

Value of portfolio $\quad P = \frac{\partial f}{\partial S} S - f(S,t)$

Risk hedge

$$P = \frac{\partial f}{\partial S} S - f(S,t)$$

$$dP = \frac{\partial f}{\partial S} dS - df(S,t) = \frac{\partial f}{\partial S} \mu S \, dt + \frac{\partial f}{\partial S} \sigma S \, dW$$

$$- \left[\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right] dt - \frac{\partial f}{\partial S} \sigma S dW$$

$$= - \left[\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right] dt$$

(A) No risk gain

Black-Scholes Eq.

Alternatively, you can increase the money "*P*" simply by a safe interest "*r*" with no risk as follows.

$$dP = rPdt = r\left(\frac{\partial f}{\partial S}S - f(S,t)\right)dt$$ (B) No risk gain

$$\therefore \quad r\left(\frac{\partial f}{\partial S}S - f(S,t)\right)dt = -\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right]dt$$

(A) = (B) No arbitrage / No free lunch

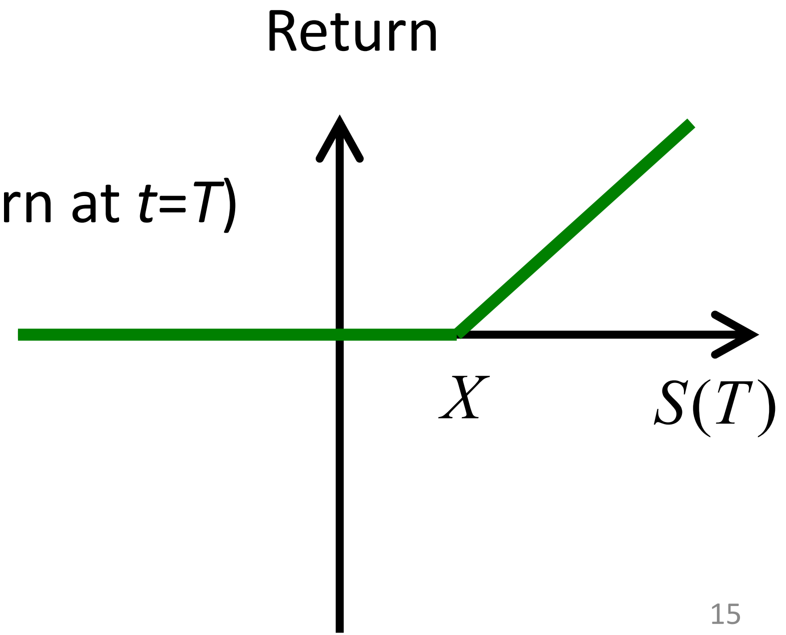
Black-Scholes Eq.

$$rf = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + r\frac{\partial f}{\partial S}S$$

Black-Scholes differential Eq.

BC for call option (premium=return at *t*=*T*)

$$f(S(T),T) = \begin{cases} S(T) - X & (S(T) \geq X) \\ 0 & (S(T) < X) \end{cases}$$



Black-Scholes Eq.

Variable conversion

$$x \equiv T - t, \quad u \equiv \ln\frac{S}{X} + \left(r - \frac{\sigma^2}{2}\right)(T - t), \quad y(u,x) \equiv e^{rx} f(S,t)$$

$$\boxed{\frac{\partial^2 y}{\partial u^2} - \frac{2}{\sigma^2}\frac{\partial y}{\partial x} = 0}$$

Diffusion Eq.

$$y(u,0) = \begin{cases} X(e^u - 1) & (u \geq 0) \\ 0 & (u < 0) \end{cases}$$

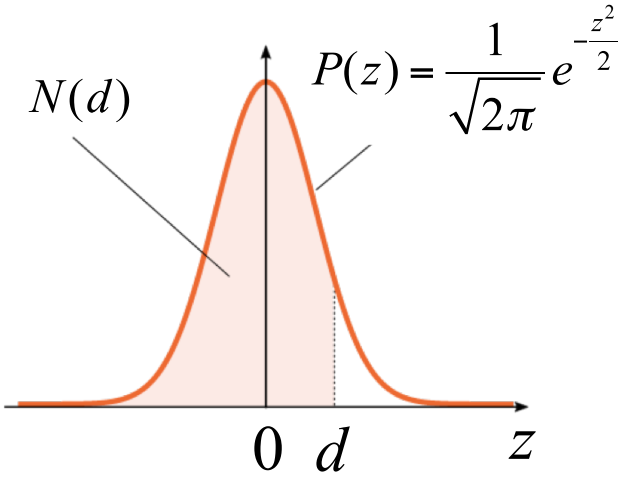
BC

Black-Scholes Eq.

Solution to the Diffusion Eq.

$$y(u,x) = Se^{rx} N\left(\frac{u}{\sigma\sqrt{x}} + \sigma\sqrt{x}\right) - XN\left(\frac{u}{\sigma\sqrt{x}}\right)$$

where $\quad N(d) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{z^2}{2}} dz$

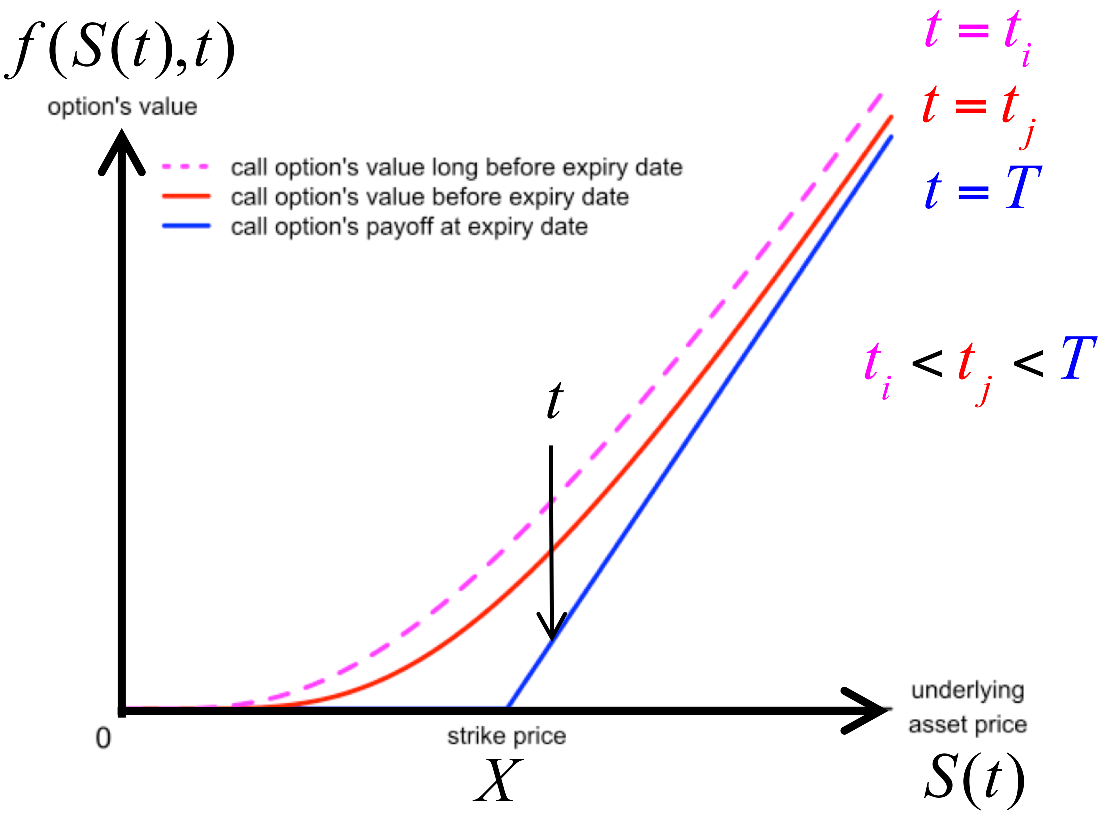


$$\therefore \boxed{f(S,t) = SN\left(\frac{u}{\sigma\sqrt{x}} + \sigma\sqrt{x}\right) - Xe^{-rx} N\left(\frac{u}{\sigma\sqrt{x}}\right)}$$

Black-Scholes Eq. (1973)
→ Nobel Prize (1997)

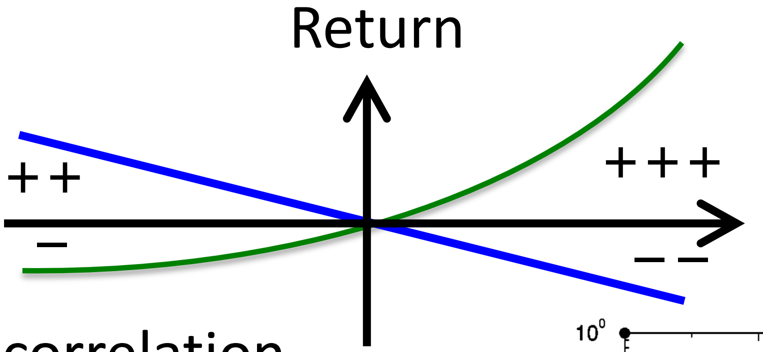
Value / Price of derivative "*f*" (=Premium) at "*t*"

Black-Scholes Eq.

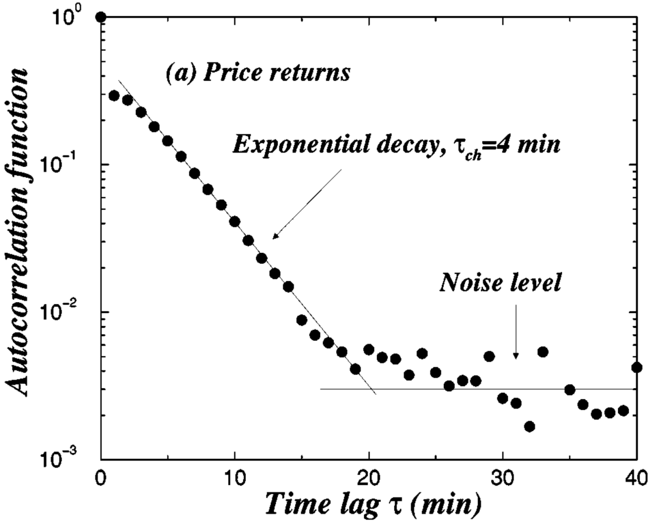


Some hints for future studies

1. Nonlinearity

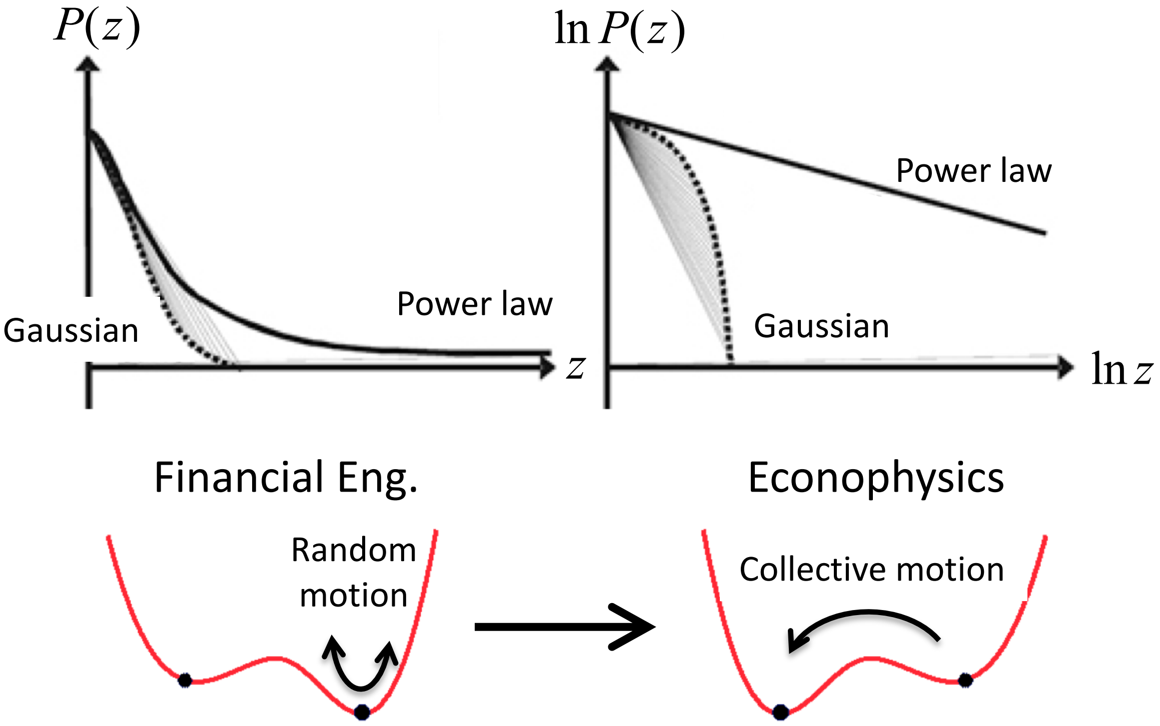


2. Auto / Cross correlation



3. Collective behavior

Financial Eng. → Econophysics



Phase transition / Phase separation

Power law in physics

- Phase transition
- Critical phenomena
- Turbulence
- Glass transition
- Fractal
- Self organization
- Geophysics
- Meteorology
- ...

Collective behavior

References

1. 石村貞夫，石村園子，「増補版　金融・証券のためのブラック・ショールズ微分方程式」（東京都書，2008）
2. 鈴木誠治，「儲かる物理」（技術評論社，2017）
3. 増川純一，水野貴之，村井浄信，尹熙元，「株価の経済物理学」（培風館，2011）
4. 松原望，「入門確率過程」（東京都書，2003）
5. J-P. Bouchaud, M. Potter, "Theory of financial risks: from statistical physics to risk management", (Cambridge, 2000)
   「金融リスクの理論」，（朝倉書店，2003）
6. M. Buchanan, "Forecast", (Bloomsbury, 2014)
   「市場は物理法則で動く」，（白楊舎，2015）